Phase velocity effects in tertiary wave interactions

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It is shown that, when two trains of waves in deep water interact, the phase velocity of each is modified by the presence of the other. The change in phase velocity is of second order and is distinct from the increase predicted by Stokes for a single wave train. When the wave trains are moving in the same direction, the increase in velocity Δc_2 of the wave with amplitude a_2, wave-number k_2 and frequency σ_2 resulting from the interaction with the wave (a_1, k_1, σ_1) is given by $\Delta c_2 = a_1^2 k_1 \sigma_1$, provided $k_1 < k_2$. If $k_1 > k_2$, then Δc_2 is given by the same expression multiplied by k_2/k_1. If the directions of propagation are opposed, the phase velocities are *decreased* by the same amount. These expressions are extended to give the increase (or decrease) in velocity due to a continuous spectrum of waves all travelling in the same (or opposite) direction.

1. Introduction

It has recently been shown (Phillips 1960) that, in the third approximation to the theory of gravity waves of small amplitude, certain unexpected effects occur. For example, there exist sets of three primary wave trains which interact to give a continuous transfer of energy to a fourth train whose amplitude increases linearly with time. More recently, Longuet-Higgins (1962) calculated explicitly the coupling constant when two of the three primary wave-numbers coincide. In these two papers, attention was directed towards situations in which the tertiary wave-number $\mathbf{k}_4$, say, is distinct from the three primary wave-numbers $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$. This is sufficient to ensure that the interaction represents a genuine energy transfer to a new component of the wave field. However, there exist cases in which the wave-number $\mathbf{k}_4$ of the tertiary wave is the same as that of one of the primary waves, say $\mathbf{k}_1$. The present note is concerned with such situations; it is shown below that tertiary waves of this kind are in quadrature with the corresponding primary wave, so that the result is not to produce a transfer of energy, but a modification of the phase velocity of the component with wave-number $\mathbf{k}_1$.

Both these effects, the energy transfer and the phase-velocity changes, can be attributed to a type of resonant interaction which occurs when the wave-numbers are related by the equation

$$\mathbf{k}_1 \pm \mathbf{k}_2 \pm \mathbf{k}_3 \pm \mathbf{k}_4 = 0, \tag{1.1}$$

and simultaneously the frequencies by

$$\sigma_1 \pm \sigma_2 \pm \sigma_3 \pm \sigma_4 = 0. \tag{1.2}$$

where the same combination of signs is to be taken in each relation and

$$\sigma_n = \{g\,|\mathbf{k}_n|\}^{\frac{1}{2}} \quad (n = 1, 2, 3, 4).$$

If the tertiary wave-number $\mathbf{k}_4$ equals $\mathbf{k}_1$, one of the primary wave-numbers, it is clear that the resonance conditions (1.1) and (1.2) can be satisfied if $\mathbf{k}_2 = \mathbf{k}_3$; that is, if the four wave-numbers are equal in pairs. The authors believe that this is the most general case which results in phase-velocity modifications.† The increase in velocity of a single wave train at the third-order approximation can be considered as the case in which *all* the wave-numbers coincide.

The following section makes use of the method developed by Longuet-Higgins (1961). In the final section, the results are extended to the situation in which a whole spectrum of parallel wave-numbers is present.

The presence of a possible second-order vorticity is ignored in this paper, but, to the third approximation, its influence on the phase velocity can be simply superimposed on the effects considered here.

2. The phase velocity effect

It will be convenient to refer directly to the equations derived by Longuet-Higgins (1961), which will be identified by the initials LH. The same notation will be used. The boundary condition at the mean free surface for the velocity potential resulting from the interaction of the two wave trains is given by equation (LH 3.11), i.e.

$$-\left(\frac{\partial^2\phi_{21}}{\partial t^2}+g\frac{\partial\phi_{21}}{\partial z}\right) = \zeta_{10}\frac{\partial}{\partial z}\left(\frac{\partial^2\phi_{11}}{\partial t^2}+g\frac{\partial\phi_{11}}{\partial z}\right)+\frac{\partial}{\partial t}(2\mathbf{u}_{11}.\mathbf{u}_{10})+\zeta_{10}\frac{\partial^2}{\partial z\,\partial t}(2\mathbf{u}_{10}.\mathbf{u}_{01})$$
$$+\mathbf{u}_{10}.\nabla(\mathbf{u}_{10}.\mathbf{u}_{01})+\mathbf{u}_{01}.\nabla(\tfrac{1}{2}\mathbf{u}_{10}^2), \tag{2.1}$$

in which the water depth is assumed to be large compared with any of the wave-lengths involved, and all terms are taken at $z = 0$. Complete expressions for the various terms on the right-hand side are given in (LH 3.12) to (LH 3.16); here we are interested in the terms proportional to $\sin\psi_2$. Omitting the others, one finds from these equations that

$$\zeta_{10}\frac{\partial}{\partial z}\left(\frac{\partial^2\phi_{11}}{\partial t^2}+g\frac{\partial\phi_{11}}{\partial z}\right) = a_1^2a_2\sigma_1\sigma_2[(\sigma_1-\sigma_2)\,|\mathbf{k}_1-\mathbf{k}_2|\cos^2\tfrac{1}{2}\theta$$
$$+(\sigma_1+\sigma_2)\,|\mathbf{k}_1+\mathbf{k}_2|\sin^2\tfrac{1}{2}\theta]\sin\psi_2, \tag{2.2}$$

$$\frac{\partial}{\partial t}(2\mathbf{u}_{11}.\mathbf{u}_{10}) = 2a_1\sigma_1\sigma_2[A\,|\mathbf{k}_1-\mathbf{k}_2|\sin^2\tfrac{1}{2}\alpha+B\,|\mathbf{k}_1+\mathbf{k}_2|\sin^2\tfrac{1}{2}\beta]\sin\psi_2, \tag{2.3}$$

$$\zeta_{10}\frac{\partial^2}{\partial z\,\partial t}(2\mathbf{u}_{10}.\mathbf{u}_{01}) = a_1^2a_2\sigma_1\sigma_2(k_1+k_2)\,(\sigma_2\cos\theta-\sigma_1)\sin\psi_2, \tag{2.4}$$

$$\mathbf{u}_{10}.\nabla(\mathbf{u}_{10}.\mathbf{u}_{01}) = a_1^2a_2\sigma_1^2\sigma_2(k_1+k_2\cos^2\tfrac{1}{2}\theta\,\sin^2\tfrac{1}{2}\theta)\sin\psi_2, \tag{2.5}$$

$$\mathbf{u}_{01}.\nabla(\tfrac{1}{2}\mathbf{u}_{10}^2) = a_1^2a_2\sigma_1^2\sigma_2k_1\sin\psi_2, \tag{2.6}$$

† It is possible to have $\mathbf{k}_4 = \mathbf{k}_1$ and $\mathbf{k}_2 \neq \mathbf{k}_3$, but in this case the phase difference between $\mathbf{k}_4$ and $\mathbf{k}_1$ is not generally $\frac{1}{2}\pi$.

where the functions A and B are given by (LH 3.10) and the angles α, β and θ are illustrated in LH figure 1. It follows that the contribution at wave-number k_2 of the interaction is given by

$$-\left(\frac{\partial^2\phi_{21}}{\partial t^2}+g\frac{\partial\phi_{21}}{\partial z}\right) = K'\sin\psi_2, \tag{2.7}$$

where

$$\begin{aligned} K' = a_1^2 a_2\sigma_1\sigma_2\Big[&(\sigma_1-\sigma_2)\,|\mathbf{k}_1-\mathbf{k}_2|\cos^2\tfrac{1}{2}\theta\left\{1+\frac{4\sigma_1\sigma_2\sin^2\frac{1}{2}\alpha}{(\sigma_1-\sigma_2)^2-g\,|\mathbf{k}_1-\mathbf{k}_2|}\right\}\\ &+(\sigma_1+\sigma_2)\,|\mathbf{k}_1+\mathbf{k}_2|\sin^2\tfrac{1}{2}\theta\left\{1-\frac{4\sigma_1\sigma_2\sin^2\frac{1}{2}\beta}{(\sigma_1+\sigma_2)^2-g\,|\mathbf{k}_1+\mathbf{k}_2|}\right\}\\ &+\sigma_1(k_1-k_2+k_2\cos^2\tfrac{1}{2}\theta\sin^2\tfrac{1}{2}\theta)+\sigma_2(k_1+k_2)\cos\theta\Big]. \end{aligned} \tag{2.8}$$

The surface displacement corresponding to this contribution is found in a manner similar to LH § 4, and

$$\zeta_{21} \sim (K't/2g)\sin\psi_2 = (K't/2g)\sin(\mathbf{k}_2.\mathbf{x}-\sigma_2 t), \tag{2.9}$$

apart from bounded third-order terms. The other components in the wave field having wave-number $\mathbf{k}_2$ are the primary wave

$$\zeta_{01} = a_2\cos\psi_2,$$

and the tertiary self-interaction term

$$\zeta_{03} = \tfrac{1}{2}a_2(a_2k_2)^2\sigma_2 t\sin\psi_2$$

(see, for example, Phillips 1960, § 5.1). Thus the total component of wave-number $\mathbf{k}_2$ is given by

$$\begin{aligned} \zeta_{01}+\zeta_{03}+\zeta_{21} &= a_2\{\cos\psi_2+[K'(2g\sigma_2a_2)^{-1}+\tfrac{1}{2}a_2^2k_2^2]\,(\sigma_2t)\sin\psi_2\}\\ &= a_2\cos\{\mathbf{k}_2.\mathbf{x}-[1+(2g\sigma_2a_2)^{-1}K'+\tfrac{1}{2}(a_2k_2)^2]\,\sigma_2t\}, \end{aligned} \tag{2.10}$$

over a sufficient time interval. The change in the phase velocity of wave 2 resulting from its interaction with wave 1 is therefore

$$\Delta c_2 = \frac{\sigma_2}{k_2}\frac{K'}{2ga_2\sigma_2} = \frac{K'}{2a_2\sigma_2^2}, \tag{2.11}$$

where the function K' is given in (2.8) above. Since K' is proportional to a_2 this change in phase velocity is thus independent of the amplitude of wave 2, whereas, of course, the additional Stokes effect is proportional to $(a_2k_2)^2$.

3. Parallel primary waves

The expression (2.8) simplifies considerably if the wave-numbers $\mathbf{k}_1$ and $\mathbf{k}_2$ are parallel. If $k_1 < k_2$ then from LH figure 1, $\theta = \alpha = 0$, $\beta = \pi$, and, from (2.8),

$$K' = a_1^2a_2\sigma_1\sigma_2[(\sigma_1-\sigma_2)(k_2-k_1)+\sigma_1(k_1-k_2)+\sigma_2(k_1+k_2)] = 2a_1^2a_2k_1\sigma_1\sigma_2^2,$$

and

$$\Delta c_2 = a_1^2k_1\sigma_1, \tag{3.1}$$

from (2.11). In this particular case, the increase in phase velocity is independent of the wave-number of wave 2 as well as its amplitude, and in fact, equals the mass-transport at the surface of wave 1. This is a curious result, and at first sight suggests that the interaction can be considered physically as a simple convection of the second wave with the mass-transport velocity induced by the first. However, such a concept is likely to be misleading, since it would lead us to expect that the distribution of mass-transport velocity with depth would be important, particularly when k_1 and k_2 are of comparable magnitude.

If $k_1 > k_2$ and the vectors are parallel, then $\theta = 0$, $\alpha = \beta = \pi$, and

$$K' = a_1^2 a_2 \sigma_1 \sigma_2 \left[(\sigma_1 - \sigma_2)(k_1 - k_2) \left\{ 1 + \frac{4\sigma_1\sigma_2}{(\sigma_1-\sigma_2)^2 - \sigma_1^2 + \sigma_2^2} \right\} + \sigma_1(k_1-k_2) + \sigma_2(k_1+k_2) \right] = 2a_1^2 a_2 k_2 \sigma_1 \sigma_2^2, \tag{3.2}$$

after a little algebra. Consequently

$$\Delta c_2 = a_1^2 \sigma_1 k_2, \tag{3.3}$$

or k_2/k_1 times the mass-transport velocity at the surface of wave 1. Clearly, in this case the idea of a simple convection fails, and the increase in c_2 is independent of the wave-number of wave 1. It can be shown likewise that, if the directions of $\mathbf{k}_1$ and $\mathbf{k}_2$ are opposite, the change Δc_2 has the magnitude (3.1) or (3.3) depending on the ratio of the wave-number magnitudes, but has the opposite sign, and so represents a *decrease* in the phase velocity of the wave trains.

4. A continuous spectrum of parallel waves

The expressions (3.1) and (3.3) can be extended readily to give the increase in velocity of a single sine wave (a_2, k_2, σ_2) resulting from the interaction with a continuous spectrum of waves all travelling in the same direction. If

$$\sum_{\sigma}^{\sigma+d\sigma} \tfrac{1}{2}a_1^2 = E(\sigma)\,d\sigma,$$

then

$$\begin{aligned}\Delta c_2 &= \int_0^{\sigma_2} 2E(\sigma)\,\sigma k\,d\sigma + \int_{\sigma_2}^{\infty} 2E(\sigma)\,\sigma k_2\,d\sigma,\\ &= \frac{2}{g}\left\{\int_0^{\sigma_2} \sigma^3 E(\sigma)\,d\sigma + \sigma_2^2 \int_{\sigma_2}^{\infty} \sigma E(\sigma)\,d\sigma\right\}.\end{aligned} \tag{4.1}$$

If the single sine wave is moving in the opposite direction to the waves in the continuous spectrum, then the velocity of the single wave *decreases* by the amount (4.1).

REFERENCES

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